NOTES ON KEISLER MEASURES

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1. Preliminaries and notation

Throughout these notes, unless otherwise specified, assume we have fixed L an arbitrary language, T a complete first-order L-theory with infinite models, and $\mathcal{U} \models T$ a monster model which is universal and strongly $\hat{\kappa}$-homogeneous for some sufficiently large cardinal $\hat{\kappa}$ (see [4, Definition 6.15 and Theorem 6.16]). We say a set is *small* if its cardinality is strictly less than $\hat{\kappa}$.

If $A \subseteq U$ is a set of parameters, we use L_A to denote the language $L \cup \{c_a : a \in A\}$ where each c_a is a constant symbol, $\mathcal{U}_A$ to denote the expansion of $\mathcal{U}$ to the L_A-structure satisfying $c_a = a$ for each $a \in A$, and T_A to denote $\mathrm{Th}\,(\mathcal{U}_A)$ the full theory of the expansion.

We frequently overload a language symbol L using it to denote the set of all L-formulae. We may restrict to formulae whose free variables have specific ranges. Let $X_0, \dots, X_{n-1}$ be named ranges (i.e., tuples of L-sorts). We use $L(X_0, \dots, X_{n-1})$ to denote the set of all L-formulae $\phi(x_0, \dots, x_{n-1})$ with free variables among $x_0, \dots, x_{n-1}$ where each x_i ranges over X_i. If we do not wish to specify the range of a variable, we use $*$ in place of a named range. For example, if X is a tuple of sorts, we write $L(X, *)$ to indicate the set of all L-formulae $\phi(x, y)$ where x ranges over X but y may have any range.

Most of the time, we do not name ranges explicitly but simply use a variable's name to refer to both the variable and its range depending on the context; for example, we write $L(x_1, \dots, x_{n-1})$ to denote the set of all L-formulae $\phi(x_0, \dots, x_{n-1})$ with free variables among $x_0, \dots, x_{n-1}$. **Any variable may be a tuple, finite or infinite, unless otherwise specified.** If L is one-sorted, then naming a range is the same as specifying a fixed tuple length. In this case, we may write $L(\kappa_0, \dots, \kappa_{n-1})$ to indicate the set of all L-formulae $\phi(x_0, \dots, x_{n-1})$ where each x_i has length κ_i. Of course, we may mix any of these conventions; for example, the expression $\phi(x, y, z) \in L(X, n, *)$ means that ϕ is an L-formula, x ranges over X, y is an n-tuple, and z may have any range.

Given $\mathcal{M} \models T$, we use M to denote the domain of $\mathcal{M}$, unless otherwise specified, and write $b \in M$ to indicate that b is a parameter from that domain. **Any parameter may be a tuple, finite or infinite, unless otherwise specified.** If L is multisorted, then we view the domain M as a disjoint union of the domains for each sort, and if X is a tuple of sorts, we write M^X to denote all tuples of parameters from M which are compatible with the range X.

1.1. Types and Type Spaces.

Let X be a tuple of L-sorts, and let $b \in U^X$. We use $\mathrm{tp}_A(b)$ to denote the complete type of b over A; i.e.,

$$\mathrm{tp}_A(b) = \{\phi \in L_A(X) : \mathcal{U} \models \phi(b)\}.$$

Given $b_1, b_2 \in U^X$, we write $b_1 \equiv_A b_2$ if b_1 and b_2 have the same complete type over A. We use $S_A(X)$ to denote the set of all X-types over A; i.e.,

$$S_A(X) = \left\{ \mathrm{tp}_A(b) \,:\, b \in U^X \right\}.$$

Of course, we may use the conventions discussed above and write $S_A(x)$ where x is a variable or $S_A(\kappa)$ where κ specifies the tuple length. If we omit the subscript A in any of the above, we mean to be working in L and T without named parameters.

The standard topology on $S_A(X)$ is totally disconnected with a clopen basis $\{[\phi]_A : \phi \in L_A(X)\}$ where $[\phi]_A$ denotes the set of all $p \in S_A(X)$ such that $\phi \in p$. If the base under consideration is clear, we often omit it, writing $[\phi]$ for $[\phi]_A$.

2. Product Spaces and Projections

Suppose $(X_\alpha, \mathcal{T}_\alpha)$ is a topological space for each $\alpha < \kappa$.

Definition 2.1. Given $A \subseteq \kappa$, we define the *projection*

$$\pi_A : \prod_{\alpha<\kappa} X_\alpha \to \prod_{\alpha \in A} X_\alpha \qquad \text{by} \qquad \prod_{\alpha<\kappa} a_\alpha \mapsto \prod_{\alpha \in A} a_\alpha.$$

In the case where $A = \{\alpha\}$ is a singleton, we simply write π_α for π_A.

Definition 2.2. We call the topology on $\prod\limits_{\alpha<\kappa} X_\alpha$ generated by the subbasis

$$\{\pi_\alpha^{-1}(G_\alpha) : \alpha < \kappa,\ G_\alpha \in \mathcal{T}_\alpha\}$$

the *product topology* and denote it as $\bigotimes\limits_{\alpha<\kappa} \mathcal{T}_\alpha$.

Notice that $\otimes$ is associative; in fact, given $A \subseteq \kappa$, we have

$$\left(\bigotimes_{\alpha \in A} \mathcal{T}_\alpha \right) \otimes \left(\bigotimes_{\beta \in \kappa \setminus A} \mathcal{T}_\beta \right) = \bigotimes_{\alpha<\kappa} \mathcal{T}_\alpha.$$

Let $X = X_0 \times X_1$ and $\mathcal{T} = \mathcal{T}_0 \otimes \mathcal{T}_1$.

Lemma 2.3. *The projection $\pi_0 : X \to X_0$ is surjective, continuous, and open. Furthermore, if $(X_1, \mathcal{T}_1)$ is compact, then π_0 is also closed.*

Proof. Surjectivity is obvious, and we can see that π_0 is an open continuous map by observing that

$$\mathcal{B} = \{G_1 \times G_2 : G_i \in \mathcal{T}_i\}$$

is a basis for $(X, \mathcal{T})$ and that both π_0 and π_0^{-1} preserve unions.

Suppose X_1 is compact and $F \subseteq X$ is closed. We need to show that $\pi_0(F)$ is closed. If $\pi_0(F) = X_0$, we are done. Otherwise, there exists $b_0 \in X_0 \setminus \pi_0(F)$, so the Tube Lemma [2, 26.8] asserts the existence of an open set $G \subseteq X_0$ such that $b_0 \in G \subseteq X_0 \setminus \pi_0(F)$. $\square$

3. The Topology of Type Space

Fix a language L and a complete L-theory T with no finite models. Let $\mathcal{U}$ be a monster model for T. Given $D \subset U$, we can think of $L_D(x)$ as the collection of subsets of U^x which are definable over D. Explicitly, if we let

$$\mathcal{H} = \{h \in 2^{L_D(x)} : \text{ if } T \vdash \phi \leftrightarrow \psi, \text{ then } h(\phi) = h(\psi)\},$$

and

$$\mathcal{D} = \{\phi(U) : \phi \in L_D(x)\},$$

then there is a natural homeomorphism $f : \mathcal{H} \to 2^{\mathcal{D}}$. See Figure 1. Furthermore, if D is small, we can think of $S_D(x)$ as the collection of atoms in the complete algebra on U generated by $\mathcal{D}$. Explicitly, if we let

$$\mathcal{D}^* = \{p(U) : p \in S_D(x)\}$$

then there is a natural bijection

$$g : f(S_D(x)) \to \mathcal{D}^*.$$

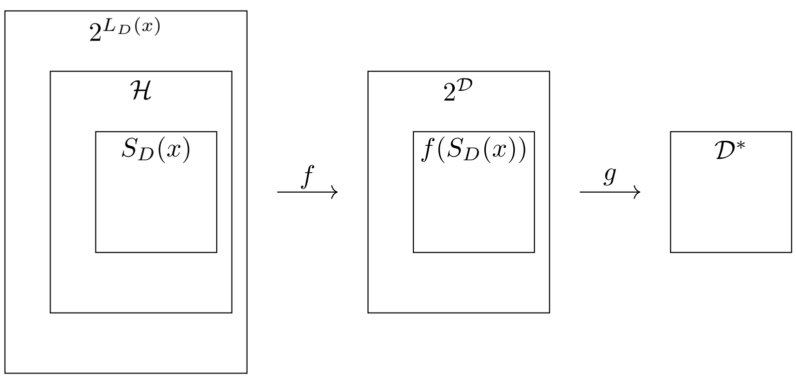

Figure 1.

Let $A \subseteq B \subseteq U$.

Proposition 3.1. *The projection $\pi : S_B(x) \to S_A(x)$, which maps each type p to its restriction*

$$p\restriction_A = \{\phi \in L_A(x) : \phi \in p\},$$

is closed, continuous, and surjective.

Proof. Given a set X, let $\mathcal{T}_X$ denote the product topology on 2^X, or more specifically, let

$$\mathcal{T}_X = \bigotimes_{x \in X} \mathcal{P}(\{0,1\}).$$

Let $D \subseteq U$. By Tychonoff [2, Theorem 37.3], the space

$$\left(2^{L_D(x)}, \mathcal{T}_{L_D(x)}\right)$$

is compact. Furthermore, note that $S_D(x)$ in the usual topology is a closed subspace and is, therefore, compact. Notice that

$$2^{L_B(x)} = 2^{L_A(x)} \times 2^{L_B(x) \setminus L_A(x)}$$

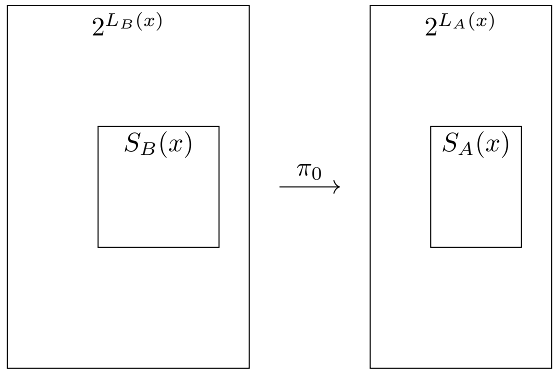

FIGURE 2.

and

$$\mathcal{T}_{L_B(x)} = \mathcal{T}_{L_A(x)} \otimes \mathcal{T}_{L_B(x) \setminus L_A(x)},$$

so Lemma 2.3 implies that

$$\pi_0 : 2^{L_B(x)} \to 2^{L_A(x)}$$

is closed and continuous. See Figure 2. Since π is the restriction of π_0 to $S_B(x)$ and since both $S_B(x)$ and $S_A(x)$ are closed, it follows that π is closed and continuous. Finally, since every type in $S_A(x)$ has an extension in $S_B(x)$, the projection π is surjective. □

For expository purposes, we provide an alternative proof of Proposition 3.1 which does not appeal to Lemma 2.3.

Proof. As we noted above, the projection map π is surjective since every type in $S_A(x)$ is a partial type in $S_B(x)$ and, therefore, has a completion in $S_B(x)$. Furthermore, the map is continuous since for all $\psi \in L_A(x)$, we have $\pi^{-1}([\psi]) = [\psi]$.

It remains to show that π is closed. Suppose $P \subseteq S_B(x)$ is closed. It follows that

$$P = \bigcap_{\phi \in \Phi} [\phi]$$

for some $\Phi \subseteq L_B(x)$. Let

$$Q = \bigcap \{[\psi] : \psi \in L_A(x),\ T + \Phi \vdash \psi\}.$$

It is easy to see that $\pi(P) \subseteq Q$. In order to prove the other inclusion, let $q \in Q$. Assume that $T + \Phi + q$ is inconsistent. By compactness, there is $\psi \in q$ such that $T + \Phi \vdash \neg\psi$. However, this implies that $\neg\psi \in q$, a contradiction. □

The projection $\pi : S_B(x) \to S_A(x)$ may not be open. For example, consider the theory of infinite sets in the empty language. Let $b \in B \setminus A$, and let $p(x) = \mathrm{tp}_B(b)$. It follows that the singleton $\{p\} = [x = b]$ is open in $S_B(x)$. However, its projection $\{p{\restriction}_A\}$ is not open in $S_A(x)$.

Let $L^* \supseteq L$ be a language. Choose a model $\mathcal{M} \models T$ containing B, and let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ to an L^*-structure. Let $T^* = \mathrm{Th}(\mathcal{M}^*)$, and let $S^*_B(x)$ denote the space of x-types over B in T^*.

Corollary 3.2. *The reduction map* $S_B^*(x) \to S_A(x)$ *given by*

$$p \mapsto \{\psi \in L_A(x) : \psi \in p\}$$

is closed, continuous, and surjective.

Proof. Modify the proof of Proposition 3.1 by replacing $L_B(x)$ and $S_B(x)$ with $L_B^*(x)$ and $S_B^*(x)$, respectively. □

Proposition 3.3. *The restriction map*

$$r : S_A(x, y) \to S_A(x)$$

given by

$$p \ \mapsto \ p\lfloor_x = \{\phi \in L_A(x) : \phi \in p\}$$

is open, closed, continuous, and surjective.

Proof. To show that r is closed, continuous, and surjective, we can modify the proof of Proposition 3.1 by replacing $L_B(x)$ and $S_B(x)$ with $L_A(x, y)$ and $S_A(x, y)$, respectively.

It remains to show that r is open. Let $\phi \in L_A(x, y)$, and let ψ be $\exists y\, \phi(x, y)$. If $p \in [\phi]$, then $\psi \in p$ since p is finitely satisfiable. Now suppose that $q \in [\psi]$. It follows that $q + \phi$ is consistent, so there exists $p \in S_A(x, y)$ such that $p \vdash q + \phi$. This demonstrates that $r([\phi]) = [\psi]$. □

4. Keisler Measures and Regular Borel Measures

We used Chapter 7 of [3] as the primary reference for much of the content of this section.

Let $A \subseteq U$. It will be helpful to think of $L_A(x)$ as the collection of all subsets of U^x which are definable over A as discussed in the previous section.

Definition 4.1. A *Keisler measure on* $L_A(x)$ is a finitely additive probability measure on the algebra of A-definable subsets of U^x; i.e., it is a map

$$\mu : L_A(x) \to [0, 1]$$

such that $\mu(x = x) = 1$ and for all $\phi, \psi \in L_A(x)$, the following hold:

(1) If ϕ and ψ are disjoint, then $\mu(\phi \vee \psi) = \mu(\phi) + \mu(\psi)$.
(2) If ϕ and ψ define the same subset of U^x, then $\mu(\phi) = \mu(\psi)$.

4.1. Lifting Keisler Measures to Borel Measures.

Let μ be a Keisler measure on $L_A(x)$, and let $\mathcal{B}$ denote the Borel subsets of $S_A(x)$. We wish to lift μ to a regular Borel measure $\tilde{\mu} : \mathcal{B} \to [0, 1]$. We start by defining $\tilde{\mu}_0$ on the clopen subsets of $S_A(x)$ so that it agrees with μ. Explicitly, for each $\phi \in L_A(x)$, we let

$$\tilde{\mu}_0([\phi]) = \mu(\phi).$$

Next, we extend to the open sets. Let $\mathcal{G}$ denote the open subsets of $S_A(x)$, and define $\tilde{\mu}_1 : \mathcal{G} \to [0, 1]$ by

$$\tilde{\mu}_1(G) = \sup\{\mu(\phi) : \phi \in L_A(x),\ [\phi] \subseteq G\}.$$

Notice that $\tilde{\mu}_1$ extends $\tilde{\mu}_0$ since μ is monotonic.

Lemma 4.2. *If* $(G_i : i < \omega) \subseteq \mathcal{G}$ *is a sequence of open sets, then*

$$\tilde{\mu}_1\left(\bigcup_{i<\omega} G_i\right) \leq \sum_{i<\omega} \tilde{\mu}_1(G_i).$$

Proof. Let $G = \bigcup_{i<\omega} G_i$, and let $\epsilon > 0$. By the definition of $\tilde{\mu}_1$, there is $\phi \in L_A(x)$ such that $[\phi] \subseteq G$ and $\mu(\phi) \geq \tilde{\mu}_1(G) - \varepsilon$. Since $[\phi]$ is compact, there is $n < \omega$ such that $[\phi] \subseteq \bigcup_{i<n} G_i$. Furthermore, since each G_i is a union of clopen sets, we can find $\theta_0, \ldots, \theta_{n-1} \in L_A(x)$ such that $[\phi] \subseteq \bigcup_{i<n}[\theta_i]$ and each $[\theta_i] \subseteq G_i$. It follows that

$$\tilde{\mu}_1(G) \leq \mu(\phi) + \varepsilon \leq \sum_{i<n} \mu(\theta_i) + \varepsilon \leq \sum_{i<n} \tilde{\mu}_1(G_i) + \varepsilon \leq \sum_{i<\omega} \tilde{\mu}_1(G_i) + \varepsilon.$$

□

Now we extend to the closed sets. Let $\mathcal{F}$ denote the collection of closed subsets of $S_A(x)$, and define $\tilde{\mu}_2 : \mathcal{F} \cup \mathcal{G} \to [0,1]$ such that it extends $\tilde{\mu}_1$ and for every $F \in \mathcal{F}$, we have

$$\tilde{\mu}_2(F) = \inf\{\mu(\phi) : \phi \in L_A(x),\ F \subseteq [\phi]\}.$$

Lemma 4.3. *Given $F \in \mathcal{F}$ and $G \in \mathcal{G}$, if $F \subseteq G$, then there is $\theta \in L_A(x)$ such that $F \subseteq [\theta] \subseteq G$.*

Proof. The result follows since F is compact and G is a union of clopen sets. □

Lemma 4.4. *Given $X, Y \in \mathcal{F} \cup \mathcal{G}$, if $X \subseteq Y$, then $\tilde{\mu}_2(X) \leq \tilde{\mu}_2(Y)$.*

Proof. If $X, Y \in \mathcal{G}$ or $X, Y \in \mathcal{F}$, then the result follows directly from the definition of $\tilde{\mu}_1$ or $\tilde{\mu}_2$, respectively. If $X \in \mathcal{F}$ and $Y \in \mathcal{G}$, then we can employ Lemma 4.3. Finally, if $X \in \mathcal{G}$ and $Y \in \mathcal{F}$, the result follows since μ is monotonic. □

Lemma 4.5. *Given $F \in \mathcal{F}$ and $G \in \mathcal{G}$, if $F \subseteq G$, then $\tilde{\mu}_2(G \setminus F) = \tilde{\mu}_2(G) - \tilde{\mu}_2(F)$.*

Proof. Let $\varepsilon > 0$. By Lemma 4.3, there is $\theta \in L_A(x)$ such that $F \subseteq [\theta] \subseteq G$, so we can choose $\phi, \psi \in L_A(x)$ such that

$$F \subseteq [\psi] \subseteq [\theta] \subseteq [\phi] \subseteq G,$$
$$\mu(\phi) \geq \tilde{\mu}_2(G) - \varepsilon,$$

and

$$\mu(\psi) \leq \tilde{\mu}_2(F) + \varepsilon.$$

It follows that

$$\tilde{\mu}_2(G \setminus F) \geq \mu(\phi \setminus \psi) = \mu(\phi) - \mu(\psi) \geq \tilde{\mu}_2(G) - \tilde{\mu}_2(F) - 2\varepsilon.$$

Now choose $\delta \in L_A(x)$ such that $[\delta] \subseteq G \setminus F$ and $\mu(\delta) \geq \tilde{\mu}_2(G \setminus F) - \varepsilon$. Since

$$\mu(\delta) + \mu(\psi) = \mu(\delta \vee \psi) + \mu(\delta \wedge \psi) \leq \tilde{\mu}_2(G) + \varepsilon,$$

it follows that

$$\tilde{\mu}_2(G \setminus F) \leq \mu(\delta) + \varepsilon \leq \tilde{\mu}_2(G) - \mu(\psi) + 2\varepsilon \leq \tilde{\mu}_2(G) - \tilde{\mu}_2(F) + 2\varepsilon.$$

□

Corollary 4.6. *If $X \in \mathcal{F} \cup \mathcal{G}$, then $\tilde{\mu}_2(X^c) = 1 - \tilde{\mu}_2(X)$.*

Lemma 4.7. *If $F_0, F_1 \in \mathcal{F}$ and $F_0 \cap F_1 = \varnothing$, then $\tilde{\mu}_2(F_0 \cup F_1) \geq \tilde{\mu}_2(F_0) + \tilde{\mu}_2(F_1)$.*

Proof. Since $F_1^c \in \mathcal{G}$, Lemma 4.3 gives us $\theta \in L_A(x)$ such that $F_0 \subseteq [\theta] \subseteq F_1^c$. Fix $\varepsilon > 0$. Choose $\psi_0 \in L_A(x)$ such that $F_0 \subseteq [\psi_0] \subseteq [\theta]$ and

$$\mu(\psi_0) \leq \tilde{\mu}_2(F_0) + \frac{\varepsilon}{2}.$$

Similarly, choose $\psi_1 \in L_A(x)$ such that $F_1 \subseteq [\psi_1] \subseteq [\neg\theta]$ and

$$\mu(\psi_1) \leq \tilde{\mu}_2(F_1) + \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} \mu(\psi_0) + \mu(\psi_1) - \mu(F_0 \cup F_1) &= \tilde{\mu}_2([\psi_0 \vee \psi_1] \setminus (F_0 \cup F_1)) \\ &\leq \tilde{\mu}_2([\psi_0] \setminus F_0) + \tilde{\mu}_2([\psi_1] \setminus F_1) \\ &\leq \varepsilon, \end{aligned}$$

so

$$\begin{aligned} \tilde{\mu}_2(F_0 \cup F_1) &\geq \mu(\psi_0) + \mu(\psi_1) - \varepsilon \\ &\geq \tilde{\mu}_2(F_0) + \tilde{\mu}_2(F_1) - \varepsilon. \end{aligned}$$

□

Definition 4.8. We call a subset $X \subseteq S_A(x)$ *regular (with respect to $\tilde{\mu}_2$)* iff: we have

$$\sup\{\tilde{\mu}_2(F) : F \in \mathcal{F},\ F \subseteq X\} = \inf\{\tilde{\mu}_2(G) : G \in \mathcal{G},\ X \subseteq G\}.$$

Lemma 4.9. *The regular subsets of $S_A(x)$ form a σ-algebra containing $\mathcal{G}$.*

Proof. Let $\mathcal{A}$ denote the regular subsets of $S_A(x)$. It is easy to see that $\mathcal{G} \subseteq \mathcal{A}$ since $\tilde{\mu}_2$ is monotonic. In order to show that $\mathcal{A}$ is closed under taking complements, let $X \in \mathcal{A}$. It follows that

$$\begin{aligned} \sup\{\tilde{\mu}_2(F) : F \in \mathcal{F},\ F \subseteq X^c\} &= \sup\{1 - \tilde{\mu}_2(F^c) : F \in \mathcal{F},\ X \subseteq F^c\} \\ &= 1 - \inf\{\tilde{\mu}_2(G) : G \in \mathcal{G},\ X \subseteq G\} \\ &= 1 - \sup\{\tilde{\mu}_2(F) : F \in \mathcal{F},\ F \subseteq X\} \\ &= \inf\{1 - \tilde{\mu}_2(F) : F \in \mathcal{F},\ F \subseteq X\} \\ &= \inf\{\tilde{\mu}_2(F^c) : F \in \mathcal{F},\ X^c \subseteq F^c\} \\ &= \inf\{\tilde{\mu}_2(G) : G \in \mathcal{G},\ X^c \subseteq G\}. \end{aligned}$$

Finally, we show that $\mathcal{A}$ is closed under taking countable unions. Let $(X_i : i < \omega) \subseteq \mathcal{A}$, and let $\varepsilon > 0$. For all $i < \omega$, there exists $F_i \in \mathcal{F}$ and $G_i \in \mathcal{G}$ such that $F_i \subseteq X_i \subseteq G_i$ and

$$\tilde{\mu}_2(G_i \setminus F_i) \leq \frac{\varepsilon}{2^{i+1}}.$$

Let $[\phi] \subseteq \bigcup_{i<\omega} G_i$ such that

$$\mu(\phi) \geq \tilde{\mu}_2\left(\bigcup_{i<\omega} G_i\right) - \varepsilon.$$

Since $[\phi]$ is compact, there exists $n < \omega$ such that $[\phi] \subseteq \bigcup_{i<n} G_i$. It follows that

$$\tilde{\mu}_2\left(\bigcup_{i<\omega} G_i\right) - \tilde{\mu}_2\left(\bigcup_{i<n} G_i\right) \leq \varepsilon,$$

so

$$\tilde{\mu}_2\left(\bigcup_{i<\omega} G_i\right) - \tilde{\mu}_2\left(\bigcup_{i<n} F_i\right) \leq \tilde{\mu}_2\left(\bigcup_{i<n} G_i\right) - \tilde{\mu}_2\left(\bigcup_{i<n} F_i\right) + \varepsilon$$

$$= \tilde{\mu}_2\left(\bigcup_{i<n} G_i \setminus \bigcup_{i<n} F_i\right) + \epsilon \leq \tilde{\mu}_2\left(\bigcup_{i<n}(G_i \setminus F_i)\right) + \varepsilon \leq 2\varepsilon.$$

□

Let $\tilde{\mu} : \mathcal{B} \to [0,1]$ be defined by

$$\tilde{\mu}(B) = \sup\{\tilde{\mu}_2(F) : F \in \mathcal{F},\ F \subseteq B\} = \inf\{\tilde{\mu}_2(G) : G \in \mathcal{G},\ B \subseteq G\}.$$

Proposition 4.10. *If $(B_i : i < \omega) \subseteq \mathcal{B}$ is a pairwise disjoint sequence of Borel sets, then*

$$\tilde{\mu}\left(\bigcup_{i<\omega} B_i\right) = \sum_{i<\omega} \tilde{\mu}(B_i).$$

Proof. Fix $\varepsilon > 0$. For each $i < \omega$, there are $G_i \in \mathcal{G}$ and $F_i \in \mathcal{F}$ such that $F_i \subseteq B_i \subseteq G_i$ and

$$\tilde{\mu}(G_i \setminus F_i) \leq \frac{\varepsilon}{2^{i+1}}.$$

Given $n < \omega$, we have

$$\sum_{i<n} \tilde{\mu}(G_i) - \varepsilon \leq \sum_{i<n} \tilde{\mu}(F_i) \leq \sum_{i<\omega} \tilde{\mu}(B_i) \leq \sum_{i<\omega} \tilde{\mu}(G_i).$$

Furthermore, since

$$\bigcup_{i<n} F_i \subseteq \bigcup_{i<\omega} B_i \subseteq \bigcup_{i<\omega} G_i,$$

we have

$$\sum_{i<n} \tilde{\mu}(F_i) \leq \tilde{\mu}\left(\bigcup_{i<\omega} B_i\right) \leq \tilde{\mu}\left(\bigcup_{i<\omega} G_i\right) \leq \sum_{i<\omega} \tilde{\mu}(G_i).$$

It follows that

$$\left|\tilde{\mu}\left(\bigcup_{i<\omega} B_i\right) - \sum_{i<\omega} \tilde{\mu}(B_i)\right| \leq \varepsilon.$$

□

We have successfully lifted $\mu : L_A(x) \to [0,1]$, an arbitrary Keisler measure, to $\tilde{\mu} : \mathcal{B} \to [0,1]$, a regular Borel measure on $S_A(x)$. In light of Lemma 4.3, it is easy to see that $\tilde{\mu}_2$ is the unique map extending $\tilde{\mu}_0$ to $\mathcal{F} \cup \mathcal{G}$ which can further be extended to a regular measure. It follows that $\tilde{\mu}$ is the unique regular measure lifting μ to $\mathcal{B}$. Since any regular Borel measure restricts to a Keisler measure, there is a one-to-one correspondence between Keisler measures on $L_A(x)$ and regular Borel measures on $S_A(x)$. Due to this, we often do not distinguish between a Keisler measure and its corresponding Borel measure.

4.2. **Changing Bases.** Let $A \subseteq B \subseteq U$, and let $\mu : L_B(x) \to [0,1]$ be a Keisler measure. It follows that the restriction of μ to $L_A(x)$ is also a Keisler measure.

Note 4.11. To ease notation, we will usually write $\mu\restriction_A$ rather than $\mu\restriction_{L_A(x)}$ to denote the restriction of μ to $L_A(x)$.

As discussed in the previous subsection, both μ and $\mu\restriction_A$ lift uniquely to regular Borel measures on $S_B(x)$ and $S_A(x)$, respectively. Let $\pi : S_B(x) \to S_A(x)$ denote the projection $p \mapsto p\restriction_A$. Proposition 3.1 asserts that π is continuous, so the preimage of a Borel set is also Borel. Given any Borel subset X of $S_A(x)$, we will use X^* denote $\pi^{-1}(X)$.

Lemma 4.12. *If X is a Borel subset of $S_A(x)$, then $\mu(X^*) = \mu\restriction_A(X)$.*

Proof. Let $\mathcal{G}_A$ and $\mathcal{F}_A$ denote the open and closed subsets of $S_A(x)$, respectively. Similarly, let $\mathcal{G}_B$ and $\mathcal{F}_B$ denote the open and closed subsets of $S_B(x)$.

We start by proving the result for open subsets. Given $G \in \mathcal{G}_A$, let

$$\Phi_G = \{\phi \in L_A(x) : [\phi]_A \subseteq G\}.$$

It follows that

$$G^* = \bigcup_{\phi \in \Phi_G} [\phi]_B.$$

If $F \in \mathcal{F}_B$ and $F \subseteq G^*$, then since F is compact, there is $\phi \in \Phi$ such that $F \subseteq [\phi]_B$. Since μ is regular, it follows that

$$\begin{aligned}
\mu(G^*) &= \sup\{\mu(F) : F \in \mathcal{F}_B,\ F \subseteq G^*\} \\
&= \sup\{\mu(\phi) : \phi \in \Phi\} \\
&= \mu\restriction_A(G).
\end{aligned}$$

The result immediately extends to closed sets since for $F \in \mathcal{F}_B$, we have $\mu(F) = 1 - \mu(F^c)$. Finally, we employ regularity to handle arbitrary Borel sets. Let $X \subseteq S_A(x)$ be Borel. It follows that

$$\begin{aligned}
\mu\restriction_A(X) &= \sup\{\mu\restriction_A(F) : F \in \mathcal{F}_A,\ F \subseteq X\} \\
&\leq \sup\{\mu(F) : F \in \mathcal{F}_B,\ F \subseteq X^*\} \\
&= \mu(X^*) \\
&= \inf\{\mu(G) : G \in \mathcal{G}_B,\ X^* \subseteq G\} \\
&\leq \inf\{\mu\restriction_A(G) : G \in \mathcal{G}_A,\ X \subseteq G\} \\
&= \mu\restriction_A(X).
\end{aligned}$$

□

4.3. **The Support of a Keisler Measure.** Let $A \subseteq U$, and let μ be a Keisler measure on $L_A(x)$. Recall that μ lifts uniquely to a regular Borel measure on $S_A(x)$.

Definition 4.13. The *support of μ* is given by

$$\operatorname{supp}(\mu) = \{p \in S_A(x) : \text{if } \phi \in p, \text{ then } \mu(\phi) > 0\}.$$

Lemma 4.14. *The support of μ is a closed subset of $S_A(x)$.*

Proof. Suppose $p \in S_A(x) \setminus \operatorname{supp}(\mu)$. It follows that $\mu(\phi) = 0$ for some $\phi \in p$, so $p \in [\phi] \in S_A(x) \setminus \operatorname{supp}(\mu)$. □

Lemma 4.15. *The support of μ has full measure; i.e., $\mu(\mathrm{supp}(\mu)) = 1$.*

Proof. Let $\Phi = \{\phi \in L_A(x) : \mu(\phi) = 0\}$, and notice that

$$S_A(x)\setminus \mathrm{supp}(\mu) = \bigcup_{\phi\in\Phi} [\phi].$$

If $F \subseteq S_A(x)\setminus \mathrm{supp}(\mu)$ is closed, then it can be covered by finitely many null sets of the form $[\phi]$ for some $\phi \in \Phi$. Therefore, by regularity, it follows that $\mu(S_A\setminus \mathrm{supp}(\mu)) = 0$. □

4.4. **Incorporating Keisler Measures into Models.** Given $\mathcal{M}$ a small model of T and X a tuple of L-sorts, let μ be Keisler measure on $L_M(X)$. It is important to notice that μ exists only in the metatheory and not in $\mathcal{M}$ (or any other model of T). We can, however, expand $\mathcal{M}$ to a model which is aware of μ using the following construction:

First, we expand L by adding a new sort R which we intend to be the domain for a model of the real numbers. Next, we add new symbols $+$, $<$, and 1 which we intend to interpret as usual on R. Finally, for every tuple of L-sorts Y and every $\phi \in L(X,Y)$, we add a new function symbol $\mu_\phi : Y \to R$ which we intend to interpret as the function $b \mapsto \mu(\phi(x, b))$. Call the new expanded language L^*, and let

$$\mathcal{M}^* = \mathcal{M} + (\mathbb{R}, +, <, 1, \mu_\phi : \phi \in L(X, *))$$

where each new symbol is interpreted as described above. Let $T^* = \mathrm{Th}(\mathcal{M}^*)$, and let $\mathcal{N}^* \succ \mathcal{M}^*$ be a monster model for T^*. It follows that the function $\mu^* : L_{U^*}(X) \to [0, 1]$ given by

$$\mu^*(\phi(x, b)) = \mathrm{st}(\mu_\phi(b))$$

is a Keisler measure extending μ. Since the L-reduct of $\mathcal{U}^*$ is a monster model for T, this shows that any Keisler measure over a model can be extended to a global Keisler measure. We will have more to say about extending Keisler measures in Subsection 5.4.

It is important to note that since $\mathcal{U}^*$ cannot discern the standard part of a nonstandard real number, the ϕ-fibers of μ^* are not actually definable in $\mathcal{U}^*$. The calculation of the standard part takes place in the metatheory. Nevertheless, incorporating a nonstandard extension of μ into $\mathcal{U}^*$ turns out to be a very useful construction.

Lemma 4.16. *Given $\phi \in L(X, Y)$ and $\varepsilon > 0$, if $\mathcal{I} = (b_i : i < \omega) \subseteq U^*$ is an indiscernible sequence and $\mu^*(\phi(x, b_i)) \geq \varepsilon$ for each $i < \omega$, then $\{\phi(x, b_i) : i < \omega\}$ is satisfiable.*

Proof. Assume not. By the Standard Lemma, we may assume that $\mathcal{I}$ is L^*-indiscernible. Let $m < \omega$ be minimal such that

$$\mu^*\left(\bigwedge_{i\leq m} \phi(x, b_i)\right) = 0.$$

For each $j < \omega$, let ψ_j be

$$\bigwedge_{mj\, \leq\, i\, <\, m(j+1)} \phi_i(x, b_i).$$

Now for some standard real number $\delta > 0$, we have $\mu^*(\psi_j) = \delta$ for each $j < \omega$. Given $n < \omega$, it follows that

$$\mu^*\left(\bigvee_{j<n} \psi_j\right) = n\delta$$

since $\mu^*(\psi_j \wedge \psi_{j'}) = 0$ for $j \neq j'$. This, however, leads to a contradiction since $\mu^*(x = x) = 1$.

□

4.5. **Applications for NIP Theories.** Let μ be a global Keisler measure on $L_U(x)$.

Lemma 4.17. *Suppose T is NIP. Given $\phi \in L(x, y)$ and $\varepsilon > 0$, there exists $n < \omega$ such that there is no finite indiscernible sequence $(b_0, \ldots, b_{2n-1}) \subseteq U^y$ such that*

$$\mu(\phi(x, b_{2i}) \triangle \phi(x, b_{2i+1})) \geq \varepsilon \tag{$*$}$$

for all $i < n$.

Proof. Assume the lemma does not hold. For each $n < \omega$, choose a finite indiscernible sequence $(b_0^n, \ldots, b_{2n-1}^n)$ such that $(*)$ holds for $i < n$. Choose $\mathcal{M}$ a small model of T containing b_i^n for $n < \omega$ and $i < 2n$. Construct $\mathcal{M}^*$, $\mathcal{U}^*$, and μ^* as in Subsection 4.4.

By compactness, there is an L-indiscernible sequence $\mathcal{J} = (d_j : j < \omega) \subseteq U^*$ such that

$$\mu^*(\phi(x, d_{2j}) \triangle \phi(x, d_{2j+1})) \geq \varepsilon$$

for each $j < \omega$, so by Lemma 4.16, there exists $a \in U^*$ such that

$$a \models \phi(x, d_{2j}) \triangle \phi(x, d_{2j+1})$$

for each $j < \omega$. This, however, leads to a contradiction since the L-reduct of $\mathcal{U}^*$ is NIP. □

Proposition 4.18. *Suppose T is NIP. Given $\phi \in L(x, y)$ and $\varepsilon > 0$, there is no sequence $\mathcal{I} = (b_i : i < \omega) \subseteq U^y$ such that $\mu(\phi(x, b_i) \triangle \phi(x, b_j)) \geq \varepsilon$ for all $i < j < \omega$.*

Proof. Assume not. Let $\mathcal{M}$ be a small model of T containing $\mathcal{I}$, and let $\mathcal{M}^*$, $\mathcal{U}^*$, and μ^* be as discussed in Subsection 4.4. By the Standard Lemma, there is an L^*-indiscernible sequence $\mathcal{J} = (b_i^* : i < \omega) \subseteq U^*$ with the same L^*-EM-type as $\mathcal{I}$. This, however, contradicts Lemma 4.17. □

Corollary 4.19. *Suppose T is NIP. Given $\phi \in L(x, y)$ and $\varepsilon > 0$, there is a finite $B \subseteq U^y$ such that for all $d \in U^y$, we have $\mu(\phi(x, b) \triangle \phi(x, d)) < \varepsilon$ for some $b \in B$.*

Proof. Let $B \subseteq U^y$ be maximal such that for all distinct $b, b' \in B$, we have

$$\mu(\phi(x, b) \triangle \phi(x, b')) \geq \varepsilon.$$

Note that Proposition 4.18 ensures that B is finite. □

5. Properties of Keisler Measures

We used Chapter 7 of [3] as the primary reference for much of the content of this section.

5.1. **Invariant and Finitely Satisfiable Measures.** Let $\mu : L_U(x) \to [0,1]$ be a global Keisler measure, and let $A \subseteq U$ be small.

Definition 5.1. We say μ is *invariant over* A iff: for all $\phi \in L(x,*)$, if $b \equiv_A b'$, then $\mu(\phi(x,b)) = \mu(\phi(x,b'))$.

Remark 5.2. When it is not important to specify a particular base, we may simply say μ is *invariant* to indicate that it is invariant over some small unnamed set. Furthermore, if ν is a Keisler measure on $L_B(x)$ where $A \subseteq B \subseteq U$, we say that ν is *invariant over* A iff: it has a global extension which is invariant over A. Although we do not restate this remark for each new property introduced in the sequel, we will follow these same conventions for other properties of Keisler measures such as finitely satisfiable, definable, generically stable, and smooth.

Lemma 5.3. *Given* $\Gamma \subseteq S_U(x)$*, if for all* $n < \omega$ *and all* $\gamma_0, \dots, \gamma_{n-1} \in \Gamma$*, we have* $\mu\left(\bigwedge_{i<n} \gamma_i\right) > 0$*, then* Γ *extends to a complete global type in* $\operatorname{supp}(\mu)$.

Proof. Let $\hat{\Gamma} \subseteq L_U(x)$ be maximal such that $\Gamma \subseteq \hat{\Gamma}$ and for all $n < \omega$ and all $\gamma_0, \dots, \gamma_{n-1} \in \hat{\Gamma}$, we have $\mu\left(\bigwedge_{i<n} \gamma_i\right) > 0$. Assume there exists $\phi \in L_U(x)$ such that neither ϕ nor $\neg\phi$ is in $\hat{\Gamma}$. It follows that there are $\alpha_0, \dots, \alpha_{m-1}$ and $\beta_0, \dots, \beta_{m-1}$ in $\hat{\Gamma}$ such that $\mu\left(\phi \wedge \bigwedge_{i<m} \alpha_i\right) = 0$ and $\mu\left(\neg\phi \wedge \bigwedge_{i<m} \beta_i\right) = 0$. However, this implies that $\mu\left(\bigwedge_{i<m} \alpha_i \wedge \bigwedge_{i<m} \beta_i\right) = 0$, a contradiction. □

Proposition 5.4. *Suppose* T *is NIP and* $\mathcal{M} \models T$*. The measure* μ *is invariant over* M *if and only if every type in* $\operatorname{supp}(\mu)$ *is invariant over* M*.*

Proof. Let $\phi \in L_M(x,y)$, and let $b, b' \in U^y$ such that $b \equiv_M b'$.

($\Leftarrow$) Suppose $\mu(\phi(x,b) \triangle \phi(x,b')) > 0$. By Lemma 5.3, there is a type in $\operatorname{supp}(\mu)$ containing the formula $\phi(x,b) \triangle \phi(x,b')$.

($\Rightarrow$) Since $b \equiv_M b'$, by [3, Lemma 5.3], there are $b_0, \dots, b_n \in U$ for some $n \geq 1$ such that $b_0 = b$, $b_n = b'$ and for each $k < n$, there is a sequence $\mathcal{I}_k = (d_i^k : i < \omega)$ which is indiscernible over M with $d_0^k = b_k$ and $d_1^k = b_{k+1}$.

Suppose μ is invariant over M, and assume $\phi(x,b) \triangle \phi(x,b') \in p$ for some $p \in \operatorname{supp}(\mu)$. It follows that $\mu(\phi(x,b) \triangle \phi(x,b')) > 0$, so $\mu(\phi(x,d_0^k) \triangle \phi(x,d_1^k)) > 0$ for some $k < n$. However, since μ is invariant over M, this contradicts Lemma 4.17. □

Definition 5.5. We say μ is *finitely satisfiable in* A iff: for all $\phi \in L_U(x)$, if $\mu(\phi) > 0$, then there is an $a \in A$ realizing ϕ.

Lemma 5.6. *If* μ *is finitely satisfiable in* A*, then it is invariant over* A*.*

Proof. Suppose μ is finitely satisfiable in A. Let $\phi \in L(x,y)$, and let $b \equiv_A b'$. Assume $\mu(\phi(x,b)) > \mu(\phi(x,b'))$. It follows that

$$\begin{aligned}\mu\left(\phi(x,b) \setminus \phi(x,b')\right) &= \mu(\phi(x,b)) - \mu\left(\phi(x,b) \wedge \phi(x,b')\right) \\ &\geq \mu(\phi(x,b)) - \mu(\phi(x,b')) > 0,\end{aligned}$$

so there is $a \in A$ such that $a \models \phi(x,b) \wedge \neg\phi(x,b')$, a contradiction. □

Proposition 5.7. *The measure* μ *is finitely satisfiable in* A *if and only if all types in* $\operatorname{supp}(\mu)$ *are finitely satisfiable in* A*.*

Proof. ($\Rightarrow$) Suppose μ is finitely satisfiable in A. Let $p \in \operatorname{supp}(\mu)$, and let $\phi_0, \dots, \phi_{n-1} \in p$. It follows that $\bigwedge_{i<n} \phi_i \in p$, so $\mu(\bigwedge_{i<n} \phi_i) > 0$. Thus, the conjunction is satisfiable in A.

($\Leftarrow$) Suppose μ is not finitely satisfiable in A. By definition, there exists $\phi \in L_U(x)$ such that $\mu(\phi) > 0$ and $\phi(A) = \varnothing$, so Lemma 5.3 asserts that some type in $\operatorname{supp}(\mu)$ contains ϕ. □

5.2. **Definable Measures.** Let $\mu : L_U(x) \to [0,1]$ be a global Keisler measure, and let $A \subseteq U$ be small.

Definition 5.8. For all $\phi(x,y) \in L(x,*)$, let

$$\mu_\phi : U^y \to [0,1]$$

be defined by

$$\mu_\phi(b) = \mu(\phi(x,b)).$$

Definition 5.9. We say μ is *definable over* A iff: for all $\phi(x,y) \in L(x,*)$ and for all closed sets $F \subseteq [0,1]$, the preimage $\mu_\phi^{-1}(F)$ is type-definable over A, i.e., there exists $\Gamma \subseteq L_A(y)$ such that for all $b \in U^y$, we have $b \models \Gamma \Leftrightarrow \mu(\phi(x,b)) \in F$.

Lemma 5.10. *If μ is definable over A, then it is invariant over A.*

Proof. Let $\phi \in L(x,y)$ and $b \in U^y$. If μ is definable over A, then there exists $\Gamma \subseteq \operatorname{tp}_A(b)$ such that for all $b' \in U^y$, we have

$$b' \models \Gamma \quad \Longleftrightarrow \quad \mu(\phi(x,b')) = \mu(\phi(x,b)).$$

□

Definition 5.11. If μ is invariant over A, then for all $\phi(x,y) \in L(x,*)$, we define

$$[\mu_\phi]_A : S_A(y) \to [0,1]$$

such that

$$[\mu_\phi]_A(\operatorname{tp}_A(b)) = \mu(\phi(x,b))$$

for all $b \in U^y$.

Lemma 5.12. *If μ is invariant over A, then the following are equivalent:*

1. *The measure μ is definable over A.*
2. *For all $\phi \in L(x,*)$, the map $[\mu_\phi]_A$ is continuous.*
3. *For all $\phi \in L(x,*)$, if D is of the form $[0,r)$, then $[\mu_\phi]_A^{-1}(D)$ is open.*
4. *For all $\phi \in L(x,*)$, if D is of the form $(r,1]$, then $[\mu_\phi]_A^{-1}(D)$ is open.*
5. *For all $\phi \in L(x,*)$, if D is of the form $[0,r]$, then $[\mu_\phi]_A^{-1}(D)$ is closed.*
6. *For all $\phi \in L(x,*)$, if D is of the form $[r,1]$, then $[\mu_\phi]_A^{-1}(D)$ is closed.*

Proof. Let $\phi(x,y) \in L(x,*)$, and let $D \subseteq [0,1]$ be closed. In $S_A(y)$, all closed sets are of the form $\bigcap_{\gamma \in \Gamma} [\gamma]$ for some $\Gamma \subseteq S_A(y)$, so (1) $\Leftrightarrow$ (2).

Notice that (3) $\Leftrightarrow$ (4) since

$$\mu(\phi(x,b)) < r \quad \Longleftrightarrow \quad \mu(\neg\phi(x,b)) > 1 - r.$$

We can now see that (2) through (4) are equivalent since the open rays form a subbasis for $[0,1]$. The rest of the needed implications follow from the fact that in any topological space, a subset is open if and only if its complement is closed. □

Corollary 5.13. *Suppose μ is definable. If μ is invariant over A, then μ is definable over A.*

Proof. Suppose μ is definable over B. The result follows since the projection $S_{AB}(x) \to S_B(x)$ is continuous and the projection $S_{AB}(x) \to S_A(x)$ is closed (see Proposition 3.1). $\square$

Definition 5.14. We say that μ is *Borel-definable* over A iff: it is invariant over A and for all $\phi \in L(x, *)$, the map $[\mu_\phi]_A$ is Borel.

Lemma 5.15. *If μ is invariant over A, then the following are equivalent:*

(1) The measure μ is Borel-definable over A.

(2) For all $\phi \in L(x, *)$, if D is of the form $[0, r)$, then $[\mu_\phi]_A^{-1}(D)$ is Borel.

(3) For all $\phi \in L(x, *)$, if D is of the form $[0, r]$, then $[\mu_\phi]_A^{-1}(D)$ is Borel.

(4) For all $\phi \in L(x, *)$, if D is of the form $[r, 1]$, then $[\mu_\phi]_A^{-1}(D)$ is Borel.

(5) For all $\phi \in L(x, *)$, if D is of the form $(r, 1]$, then $[\mu_\phi]_A^{-1}(D)$ is Borel.

Proof. The rays of any one type generate the Borel σ-algebra on $[0, 1]$. $\square$

Lemma 5.16. *Suppose T is NIP. If a global type $p \in S_U(x)$ is invariant over A, then it is Borel-definable over A.*

Proof. Given $\phi(x, y) \in L(x, *)$, we need to show that

$$Q = \{\mathrm{tp}_A(b) : \phi(x, b) \in p\} \subseteq S_A(y)$$

is a Borel set. Let $m = \max\{\mathrm{alt}(\phi(x, b)) : b \in U\}$, and for every $n \leq m$, let

$$F_n = \bigcap_{\psi \,\in\, p^{n+1}|_A} [\psi(x_0, \dots, x_n)] \cap \bigcap_{i<n} [\phi(x_i, y) \triangle \phi(x_{i+1}, y)] \;\subseteq\; S_A(x_0, \dots, x_n, y).$$

It follows that $F_n \cap [\phi(x_n, y)]$ is closed in $S_A(x_0, \dots, x_n, y)$, so its restriction to $S_A(y)$, call it Q_n, is closed by Corollary 3.3. Similarly, the restriction of $F_n \cap [\neg\phi(x_n, y)]$ to $S_A(y)$, call it R_n, is also closed. It follows that $Q = \bigcup_{n<m}(Q_n \cap R_n^c)$, so Q is Borel. $\square$

5.3. **Generically Stable Measures.** Let $\mu : L_U(x) \to [0, 1]$ be a global Keisler measure, and let $A \subseteq U$ be small.

Definition 5.17. We say μ is *generically stable over A* iff: it is finitely satisfiable and definable, both with respect to A.

In order to give some examples of generically stable measures, we define two notions of average measure.

Definition 5.18. Given a finite sequence $\bar{a} = (a_0, \dots, a_{n-1}) \subseteq U^x$, the map

$$\mathrm{Av}_{\bar{a}} : L_U(x) \to [0, 1]$$

given by

$$\mathrm{Av}_{\bar{a}}(\phi) = \frac{1}{n} \sum_{i<n} \delta_{a_i}(\phi(U)),$$

where δ_{a_i} denotes the Dirac measure at a_i, is a Keisler measure which we call the *average measure determined by $\bar{a}$*.

It is easy to see that $\mathrm{Av}_{\bar{a}}$ is generically stable over $\{\mathrm{im}(a_i) : i < n\}$. For our second notion of average measure, we require T to be NIP.

Definition 5.19. If T is NIP and $\mathcal{I} = (b_i : i \in [0,1]) \subseteq U^x$ is indiscernible, then the map $\mathrm{Av}_{\mathcal{I}} : L_U(x) \to [0,1]$ given by

$$\mathrm{Av}_{\mathcal{I}}(\phi) = m(\{i \in [0,1] : \mathcal{U} \models \phi(b_i)\}),$$

where m denotes the Lebesgue measure, is a Keisler measure which we call the *average measure determined by* $\mathcal{I}$.

Notice that since we require T to be NIP and $\mathcal{I}$ to be indiscernible, the set $\{i \in [0,1] : \mathcal{U} \models \phi(b_i)\}$ is Lebesgue measurable for all $\phi \in L_U(x)$. In Lemma 5.21 below, we will show that $\mathrm{Av}_{\mathcal{I}}$ is generically stable over $\mathcal{I}$.

Lemma 5.20. *Suppose T is NIP. Given $\phi \in L(x,y)$ and $\varepsilon > 0$, there exists $n < \omega$ such that for every indiscernible sequence of x-tuples $\mathcal{I} = (b_i : i \in [0,1])$, there are $a_0, \ldots, a_{n-1} \in \mathcal{I}$ such that*

$$|Av_{\mathcal{I}}(\phi(x,d)) - \mathrm{Av}_{\bar{a}}(\phi(x,d))| \leq \varepsilon$$

for all $d \in U$.

Proof. Let $m = \max\{\mathrm{alt}(\phi(x,d)) : d \in U\}$, and let $n \geq m/\varepsilon$. Given an indiscernible sequence $\mathcal{I} = (b_i : i \in [0,1]) \subseteq U^x$, let $a_k = b_{k/n}$ for each $k < n$. Fix $d \in U$. Since the set

$$K = \left\{k < n \ : \ \mathcal{U} \models \phi(a_k, d) \triangle \phi(b_i, d) \text{ for some } i \in \left(\frac{k}{n}, \frac{k+1}{n}\right)\right\}$$

contains at most m elements, it follows that

$$|\,\mathrm{Av}_{\mathcal{I}}(\phi(x,d)) - \mathrm{Av}_{\bar{a}}(\phi(x,d))| \leq \frac{|K|}{n} \leq \varepsilon.$$

$\square$

Lemma 5.21. *Suppose T is NIP. If $\mathcal{I} \subseteq U$ is an indiscernible sequence of tuples indexed by $[0,1]$, then $\mathrm{Av}_{\mathcal{I}}$ is generically stable over $\mathcal{I}$.*

Proof. It is easy to see that $\mathrm{Av}_{\mathcal{I}}$ is finitely satisfiable in $\mathcal{I}$. Fix $\phi \in L(x,y)$ and $r > 0$. Let

$$Q = \{q \in S_{\mathcal{I}}(y) \ : \ \mathrm{Av}_{\mathcal{I}}(\phi(x,b)) < r \text{ for some } b \models q\}$$

Suppose $q \in Q$. Let $b \models q$, and let

$$\varepsilon = \frac{r - \mathrm{Av}_{\mathcal{I}}(\phi(x,b))}{2}.$$

By Lemma 5.20, there are $a_0, \ldots, a_{n-1} \in \mathcal{I}$ such that for all $d \in U$, we have

$$|\,\mathrm{Av}_{\mathcal{I}}(\phi(x,d)) - \mathrm{Av}_{\bar{a}}(\phi(x,d))| \leq \varepsilon.$$

Let ψ denote the conjunction

$$\bigwedge_{\mathcal{U} \models \neg\phi(a_i,b)} \neg\phi(a_i, y).$$

Notice that $q \in [\psi] \subseteq Q$, so Q is open in $S_{\mathcal{I}}(y)$. Thus, Lemma 5.12(3) asserts that $\mathrm{Av}_{\mathcal{I}}$ is definable over $\mathcal{I}$. $\square$

5.4. **Extending Measures.** Let $\mathcal{D} \subseteq L_U(x)$ be an algebra (see [1, p. 21]) of definable sets, and let $\nu : \mathcal{D} \to [0,1]$ be a finitely additive probability measure. Fix $\phi \in L_U(x)$ and $r \in [0,1]$.

Lemma 5.22. *If for all finite $\mathcal{F} \subseteq L_U(x)$, there is a map $\mu_{\mathcal{F}} : \mathcal{F} \to [0,1]$ such that*

(1) for all $D \in \mathcal{D} \cap \mathcal{F}$, we have $\mu_{\mathcal{F}}(D) = \nu(D)$,
(2) for all disjoint $F_0, F_1 \in \mathcal{F}$, if $F_0 \cup F_1 \in \mathcal{F}$, then

$$\mu_{\mathcal{F}}(F_0 \cup F_1) = \mu_{\mathcal{F}}(F_0) + \mu_{\mathcal{F}}(F_1), \text{ and}$$

(3) if $\phi \in \mathcal{F}$, then $\mu_{\mathcal{F}}(\phi) = r$,

then there is a global Keisler measure $\mu : L_U(x) \to [0,1]$ extending ν such that $\mu(\phi) = r$.

Proof. Let $X = [0,1]^{L_U(x)}$. By Tychonoff's Theorem, X is compact. For all finite $\mathcal{F} \subseteq L_U(x)$, let

$$K(\mathcal{F}) = \{\mu \in X : \mu\restriction_{\mathcal{F}} \text{ satisfies (1), (2), and (3)}\}.$$

Notice that each $K(\mathcal{F})$ is closed and, therefore, compact. Let

$$\mathcal{K} = \{K(\mathcal{F}) : \mathcal{F} \subseteq L_U(x) \text{ is finite}\}$$

Suppose that for all finite $\mathcal{F} \subseteq L_U(x)$, there is a map $\mu_{\mathcal{F}} : \mathcal{F} \to [0,1]$ such that (1), (2), and (3) hold. It follows that $\mathcal{K}$ has the finite intersection property, since

$$\bigcap_{i<n} K(\mathcal{F}_i) \supseteq K\left(\bigcup_{i<n} \mathcal{F}_i\right).$$

Furthermore, since X is compact, there exists $\mu \in \bigcap_{K\in\mathcal{K}} K$. $\square$

Proposition 5.23. *If*

$$\sup\{\nu(D) : D \in \mathcal{D},\ D \subseteq \phi\} \le r \le \inf\{\nu(D) : D \in \mathcal{D},\ \phi \subseteq D\},$$

then there is a global Keisler measure $\mu : L_U(x) \to [0,1]$ extending ν such that $\mu(\phi) = r$.

Proof. Let $\mathcal{F} \subseteq L_U(x)$ be a finite algebra of definable sets, and let $\mathcal{B} = (B_0, \dots, B_{n-1})$ be an enumeration of the atoms in $\mathcal{D} \cap \mathcal{F}$. Notice that $n \ge 1$ since all algebras on X contain X. We can enumerate the atoms of $\mathcal{F}$ as

$$\mathcal{A} = \left\{A_i^j : i < n,\ j \le m_i\right\}$$

where each $A_i^j \subseteq B_i$.

Case 1: If $\phi \notin \mathcal{F}$, then for each $i < n$ and $j \le m_i$, let

$$\mu_{\mathcal{F}}(A_i^j) = \begin{cases} \nu(B_i) & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Case 2: If $\phi \in \mathcal{F}$, we may assume we have enumerated $\mathcal{B}$ so that there exists $k < l < n$ such that

$$B_i \subseteq \phi \iff i < k \qquad \text{and} \qquad B_i \cap \phi \ne \varnothing \iff i < l.$$

Furthermore, we may assume we have enumerated $\mathcal{A}$ so that

$$i < l \implies A_i^0 \subseteq \phi \qquad \text{and} \qquad i \ge k \implies A_i^{m_i} \cap \phi = \varnothing.$$

If $i < k$, let

$$\mu_{\mathcal{F}}(A_i^j) = \begin{cases} \nu(B_i) & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $k \leq i < l$, let

$$\mu_{\mathcal{F}}(A_i^0) = \max\{\min\{r - \nu(B_0 \cup \cdots \cup B_{i-1}),\ \nu(B_i)\},\ 0\}$$

and

$$\mu_{\mathcal{F}}(A_i^j) = \begin{cases} 0 & \text{if } 0 < j < m_i, \\ \nu(B_i) - \mu_{\mathcal{F}}(A_i^0) & \text{if } j = m_i. \end{cases}$$

If $i \geq l$, let

$$\mu_{\mathcal{F}}(A_i^j) = \begin{cases} \nu(B_i) & \text{if } j = m_i, \\ 0 & \text{otherwise.} \end{cases}$$

If $F \in \mathcal{F} \setminus \mathcal{A}$, then let

$$\mu_{\mathcal{F}}(F) = \sum_{\{A \in \mathcal{A} \,:\, A \subseteq F\}} \mu_{\mathcal{F}}(A).$$

We may now appeal to Lemma 5.22. □

5.5. **Smooth Measures.** Let $\mu : L_U(x) \to [0,1]$ be a global Keisler measure, and let $A \subseteq U$ be small.

Definition 5.24. We say μ is *smooth over* A iff: it is the unique global Keisler measure extending $\mu\lfloor_A$.

Lemma 5.25. *The measure μ is smooth over A if and only if for all $\phi(x, y) \in L(x, *)$ and all $\varepsilon > 0$, there exists $n < \omega$, along with formulae*

$$\alpha_0, \ldots, \alpha_{n-1} \in L_A(x), \quad \beta_0, \ldots, \beta_{n-1} \in L_A(x), \quad \textit{and} \quad \gamma_0, \ldots, \gamma_{n-1} \in L_A(y),$$

such that

(1) $\mathcal{U} \models \forall y \bigvee_{i<n} \gamma_i(y)$,

(2) $\mathcal{U} \models \forall y \bigwedge_{i<n} [\gamma_i(y) \to \forall x\, [\alpha_i(x) \to \phi(x, y) \to \beta_i(x)]]$, *and*

(3) $\mu(\beta_i) - \mu(\alpha_i) < \varepsilon$ *for all* $i < n$.

Proof. Fix $\phi(x, y) \in L(x, *)$.

($\Rightarrow$) Suppose μ is smooth over A, and let $\varepsilon > 0$. Proposition 5.23 implies that for every $b \in U^y$ there exists $\alpha_b, \beta_b \in L_A(x)$ such that

$$\mathcal{U} \models \forall x\ [\alpha_b(x) \to \phi(x, b) \to \beta_b(x)]$$

and $\mu(\beta_b) - \mu(\alpha_b) < \epsilon$. For every $b \in U$, let ψ_b be the $L_A(y)$ formula

$$\forall x\, [\alpha_b(x) \to \phi(x, y) \to \beta_b(x)].$$

Since $\mathcal{U} \models \psi_b(b)$ for all $b \in \mathcal{U}$, we conclude that $\{\neg\psi_b : b \in U\}$ is inconsistent with T_A. Therefore, there exist $b_0, \ldots, b_{n-1} \in U$ such that $\mathcal{U} \models \forall y \bigvee_{i<n} \psi_{b_i}(y)$. For each $i < n$, let γ_i be the $L_A(y)$ formula

$$\forall x\, [\alpha_{b_i}(x) \to \phi(x, y) \to \beta_{b_i}(x)].$$

($\Leftarrow$) Given $b \in \mathcal{U}^y$, construct sequences $(\alpha_k)_{k<\omega}$ and $(\beta_k)_{k<\omega}$ in $L_A(x)$ such that for each $k < \omega$, we have

$$\mathcal{U} \models \forall x \; [\alpha_k(x) \to \phi(x, b) \to \beta_k(x)]$$

and

$$\mu(\beta_k) - \mu(\alpha_k) < 1/k.$$

$\square$

Proposition 5.26. *Suppose T is NIP. There is a global Keisler measure ν on $L_U(x)$ extending $\mu\lfloor_A$ which is smooth over some small $B \subseteq U$.*

Proof. Let $\kappa = |T|^+$. If the proposition does not hold, we can construct a chain of Keisler measures $(\nu_\alpha : \alpha < \kappa)$ extending $\mu\lfloor_A$ as follows:

$\alpha = 0$: Let $\nu_0 = \mu\lfloor_A$.

α limit: Let $\nu_\alpha = \bigcup_{\beta<\alpha} \nu_\beta$.

$\alpha + 1$: Given ν_α a Keisler measure on $L_{Ab_{<\alpha}}(x)$, choose

- $\phi_\alpha(x, y) \in L(x, *)$,
- $b_\alpha \in U^y$,
- $n_\alpha \in \mathbb{N}$, and
- Keisler measures ρ_0 and ρ_1 on $L_{Ab_{\leq\alpha}}(x)$ extending ν_α

such that $(\rho_1 - \rho_0)(\phi_\alpha(x, b_\alpha)) > 1/n_\alpha$, and let $\nu_{\alpha+1} = \frac{1}{2}(\rho_0 + \rho_1)$. Notice that for every $\theta \in L_{Ab_{<\alpha}}(x)$, if we let ϕ denote $\phi_\alpha(x, b_\alpha)$, we have

$$\rho_i(\phi \bigtriangleup \theta) = \rho_i(\phi) + \rho_i(\theta) - 2\rho_i(\phi \wedge \theta) \geq |\rho_i(\phi) - \rho_i(\theta)|,$$

and since

$$\rho_0(\theta) = \rho_1(\theta),$$

it follows that

$$\nu_{\alpha+1}(\phi \bigtriangleup \theta) \geq \frac{1}{2}|\rho_1(\phi) - \rho_0(\phi)| > \frac{1}{2n_\alpha}.$$

For each $\phi(x, y) \in L(x, *)$, let $A_\phi = \{\alpha < \kappa : \phi_\alpha \text{ is } \phi\}$. Since $\kappa = |T|^+$, there is a formula ϕ such that $|A_\phi| = \kappa$. For each $n > 0$, let $B_n = \{\alpha \in A_\phi : n_\alpha = n\}$. Again, there is a positive integer n such that $|B_n| = \kappa$. This, however, contradicts Proposition 4.18. $\square$

Lemma 5.27. *Given $\mathcal{M} \models T$, if μ is smooth over M, then it is generically stable over M.*

Proof. Given $\phi \in L_U(x)$ with $\mu(\phi) > 0$, by Lemma 5.25, there is $\alpha \in L_M(x)$ such that $\mathcal{U} \models \alpha \to \phi$ and $\mu(\alpha) > 0$. It follows that $\alpha(M) \neq \varnothing$, so μ is finitely satisfiable in M.

It remains to show that μ is definable over M. Let $\phi(x, y) \in L(x, *)$ and $b \in U^y$. Suppose $\mu(\phi(x, b)) < r$ for some $r \in [0, 1]$. By Lemma 5.25, there is $\gamma \in \mathrm{tp}_M(b)$ such that $[\gamma] \subseteq [\mu_\phi]_M^{-1}([0, r))$. Now we may appeal to Lemma 5.12(3). $\square$

Lemma 5.28. *Given $\mathcal{M} \models T$, if μ is invariant over M and smooth (over some small set), then it is smooth over M.*

Proof. Suppose μ is invariant over M and smooth. Lemma 5.27 asserts that μ is definable, so by Corollary 5.13, it is definable over M.

Fix $\phi \in L(x, y)$ and $\varepsilon > 0$. Applying Lemma 5.25 to ϕ and ε gives us a finite sequence of formulae

$$(\alpha_i(x, b),\ \beta_i(x, b),\ \gamma_i(y, b))_{i<n}$$

where $\alpha_i(x, z), \beta_i(x, z), \gamma_i(y, z) \in L$ and $b \in U^z$ such that

$$\mathcal{U} \models \bigvee_{i<n} \gamma_i(y, b) \wedge \bigwedge_{i<n} [\gamma_i(y, b) \to [\alpha_i(x, b) \to \phi(x, y) \to \beta_i(x, b)]]$$

and for each $i < n$, we have $\mu(\beta_i(x, b) \setminus \alpha_i(x, b)) < \varepsilon$. Since μ is definable over M, Lemma 5.12(3) ensures the existence of $\psi_i(z) \in \mathrm{tp}_M(b)$ for each $i < n$ such that for all $b' \in \psi_i(U)$, we have $\mu\left(\beta_i(x, b') \setminus \alpha_i(x, b')\right) < \varepsilon$. It follows that

$$\mathcal{M} \models \exists z \left[\bigwedge_{i<n} \psi_i(z) \wedge \bigvee_{i<n} \gamma_i(y, z) \wedge \bigwedge_{i<n} [\gamma_i(y, z) \to [\alpha_i(x, z) \to \phi(x, y) \to \beta_i(x, z)]]\right],$$

so Lemma 5.25 asserts that μ is smooth over M. $\square$

5.6. **Approximating Smooth Measures.** Let $\mu : L_U(x) \to [0, 1]$ be a global Keisler measure, and let $D \subseteq U$ be a small set of parameters.

Proposition 5.29. *Suppose μ is smooth over D. Given $\phi(x, y) \in L(x, *)$ and $\varepsilon > 0$, there is a finite sequence $\bar{a} = (a_0, \ldots, a_{n-1}) \subseteq U^x$ such that for all $b \in U^y$, we have*

$$|\mu(\phi(x, b)) - \mathrm{Av}_{\bar{a}}(\phi(x, b))| < \epsilon.$$

Furthermore, if $D \subseteq \mathcal{M} \models T$, then we may choose $\bar{a}$ to be a sequence in M.

Proof. Lemma 5.25 gives us a finite sequence of L_D formulae

$$(\alpha_i(x),\ \beta_i(x),\ \gamma_i(y))_{i<n}$$

such that

$$\mathcal{U} \models \bigvee_{i<n} \gamma_i(y) \wedge \bigwedge_{i<n} [\gamma_i(y) \to [\alpha_i(x) \to \phi(x, y) \to \beta_i(x)]]$$

and $\mu(\beta_i \setminus \alpha_i) < \varepsilon/2$ for each $i < n$. Let $\mathcal{B} \subseteq \mathcal{P}(U)$ be the boolean algebra generated by

$$\{\alpha_i(U), \beta_i(U) : i < n\},$$

and let $\mathcal{A}$ denote the set of atoms in $\mathcal{B}$. Fix $K > 2|\mathcal{A}|/\varepsilon$. For each $A \in \mathcal{A}$, choose $a_A \in A$ and

$$k_A \in \{\lfloor K \cdot \mu(A) \rfloor, \lceil K \cdot \mu(A) \rceil\}$$

in such a way that

$$K = \sum_{A \in \mathcal{A}} k_A.$$

Let

$$\lambda = \frac{1}{K} \sum_{A \in \mathcal{A}} k_A \delta_{a_A}.$$

It follows that λ is a Keisler measure on $L_U(x)$. Notice that for all $A \in \mathcal{A}$, we have

$$|\mu(A) - \lambda(A)| < \frac{1}{K},$$

so for all $B \in \mathcal{B}$, we have

$$|\mu(B) - \lambda(B)| < \frac{\varepsilon}{2}.$$

Given $b \in U$, fix $i < n$ such that $\mathcal{U} \models \gamma_i(b)$. Let $B_0 = \alpha_i(U)$, $B_1 = \beta_i(U)$, and $\Phi = \phi(U, b)$. Recall that $B_0 \subseteq \Phi \subseteq B_1$ and

$$\mu(B_1) - \mu(B_0) < \frac{\varepsilon}{2}.$$

It follows that

$$\lambda(B_1) - \mu(B_0) < \varepsilon$$

and

$$\mu(B_1) - \lambda(B_0) < \varepsilon,$$

so we conclude that

$$|\mu(\Phi) - \lambda(\Phi)| < \varepsilon.$$

□

Let $\pi : S_U(x) \to S_D(x)$ be the projection $p \mapsto p{\restriction}_D$. Given any Borel subset X of $S_D(x)$, we will use X^* to denote $\pi^{-1}(X)$. Recall Lemma 4.12 asserts that $\mu(X^*) = \mu{\restriction}_D(X)$.

Proposition 5.30. *Suppose μ is smooth over D. Given*

- $\phi(x, y) \in L(x, *)$,
- $\varepsilon > 0$, *and*
- $X_0, \dots, X_{m-1}$ *Borel subsets of* $S_D(x)$,

there is a finite sequence $\bar{p} = (p_0, \dots, p_{n-1}) \subseteq \operatorname{supp}(\mu)$ such that for all $b \in U^y$ and all $\ell < m$, we have

$$|\mu([\phi(x, b)]_U \cap X_\ell^*) - \operatorname{Av}_{\bar{p}}([\phi(x, b)]_U \cap X_\ell^*)| < \epsilon.$$

Proof. Lemma 5.25 gives us a finite sequence of L_D formulae

$$(\alpha_i(x),\ \beta_i(x),\ \gamma_i(y))_{i<n}$$

such that

$$\mathcal{U} \models \bigvee_{i<n} \gamma_i(y) \wedge \bigwedge_{i<n} [\gamma_i(y) \to [\alpha_i(x) \to \phi(x, y) \to \beta_i(x)]]$$

and $\mu(\beta_i \setminus \alpha_i) < \varepsilon/2$ for each $i < n$. Let $\mathcal{B} \subseteq \mathcal{P}(S_D(x))$ be the boolean algebra generated by

$$\{X_\ell : \ell < m\} \cup \{[\alpha_i(x)]_D, [\beta_i(x)]_D : i < n\},$$

and let $\mathcal{A}$ denote the set of atoms in $\mathcal{B}$. Fix $K > 2|\mathcal{A}|/\varepsilon$. For each $A \in \mathcal{A}$, choose $p_A \in A^*$ and

$$k_A \in \{\lfloor K \cdot \mu(A) \rfloor, \lceil K \cdot \mu(A) \rceil\}$$

in such a way that

$$K = \sum_{A \in \mathcal{A}} k_A$$

and if $k_A > 0$, then $p_A \in \operatorname{supp}(\mu)$. Let

$$\lambda = \frac{1}{K} \sum_{A \in \mathcal{A}} k_A p_A.$$

It follows that λ is a Keisler measure on $L_U(x)$ which extends uniquely to a regular Borel measure on $S_U(x)$, namely

$$\frac{1}{K}\sum_{A\in\mathcal{A}} k_A \delta_{p_A}.$$

Notice that for all $A \in \mathcal{A}$, we have

$$|\mu(A^*) - \lambda(A^*)| < \frac{1}{K},$$

so for all $B \in \mathcal{B}$, we have

$$|\mu(B^*) - \lambda(B^*)| < \frac{\varepsilon}{2}.$$

Given $b \in U$ and $\ell < m$, fix $i < n$ such that $\mathcal{U} \models \gamma_i(b)$. Let $B_0 = [\alpha_i(x)]_D \cap X_\ell$ and $B_1 = [\beta_i(x)]_D \cap X_\ell$. Let $\Phi = [\phi(x,b)]_U \cap X_\ell^*$. Recall that $B_0^* \subseteq \Phi \subseteq B_1^*$ and

$$\mu(B_1^*) - \mu(B_0^*) < \frac{\varepsilon}{2}.$$

It follows that

$$\lambda(B_1^*) - \mu(B_0^*) < \varepsilon$$

and

$$\mu(B_1^*) - \lambda(B_0^*) < \varepsilon,$$

so we conclude that

$$|\mu(\Phi) - \lambda(\Phi)| < \varepsilon.$$

□

Corollary 5.31. *Suppose T is NIP. Given $\phi(x,y) \in L(x,*)$ and $\varepsilon > 0$, there is a finite sequence $\bar{a} = (a_0,\ldots,a_{n-1}) \subseteq U^x$ such that for all $d \in D^y$, we have*

$$|\mu(\phi(x,d)) - \mathrm{Av}_{\bar{a}}(\phi(x,d))| < \epsilon.$$

Furthermore, if $X_0,\ldots,X_{m-1}$ are Borel subsets of $S_D(x)$, there is a finite sequence $\bar{p} = (p_0,\ldots,p_{n-1}) \subseteq S_U(x)$ such that for all $d \in D^y$ and all $\ell < m$, we have

$$|\mu([\phi(x,d)]_U \cap X_\ell^*) - \mathrm{Av}_{\bar{p}}([\phi(x,d)]_U \cap X_\ell^*)| < \epsilon.$$

Proof. By Proposition 5.26, the restriction $\mu{\restriction}_D$ extends to a global Keisler measure ν which is smooth over some small $D' \supseteq D$. □

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